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THE COMPUTER AS MASTER MIND

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A deductive game called Master Mind, produced by Invicta Plastics, Ltd., enjoyed a wave of popularity during the 1975 Christmas season [2]. In this game one player (the "codemaker") conceals a four-symbol code, and the other player (the "codebreaker") tries to identify all four symbols by trying appropriate test patterns. There are six symbols, represented by pegs of different colors, and repetitions are permitted, hence there are $6^4 = 1296$ possible codewords. If the codemaker's secret codeword is $x_1x_2x_3x_4$ and if the codebreaker gives the test pattern $y_1y_2y_3y_4$, the codemaker tells him how close he is by announcing

1. the number of "black hits," i.e., the number of positions j such that $x_j = y_j$;
2. the number of "white hits," i.e., the number of positions j such that $x_j \neq y_j$ but $x_j = y_k$ for some k and y_k has not been used in another hit.

For example, let the symbols be denoted by 1, 2, 3, 4, 5, 6; if the codeword is 2532 and the test pattern is 3523, there are two white hits and one black hit. (Rule 2 is somewhat difficult to state precisely and unambiguously, and the manufacturers have *not* succeeded in doing so on the directions they furnish with the game, although *Games and Puzzles* magazine [7] has given them a rating of 8 points out of 8 for clarity of explanation! Perhaps the clearest way to state the rule exactly, when speaking to mathematicians or to computers, is this: Let n_i be the numer of times symbol i occurs in the codeword, and n_i' the number of times it occurs in the test pattern, for $1 \leqslant i \leqslant 6$; then the number of total number of hits, both white and black, is

$$\min(n_1, n_1') + \min(n_2,n_2') + \ldots + \min(n_6,n_6').$$

It follows that the total number of misses is

$$\max(n_1-n_1',0) + \max(n_2-n_2',0) + \ldots + \max(n_6-n_6',0).$$

Note that it is impossible to have three black hits and one white hit.) The codebreaker tries to get *four black hits,* after constructing a minimum number of test patterns.

The purpose of this note is to prove that *the codebreaker can always succeed in five moves or less.* Thus, he knows the code after at most four guesses; and he often discovers it earlier. A complex strategy for doing this, worked out with the aid of a computer, appears in Figure 1 in highly condensed form. Readers who have played Master Mind will be able to see how a real expert operates by studying this strategy carefully.

Here is how to read Figure 1: Every situation we can arrive at during a game corresponds to a certain number of codewords, and it is assumed that the reader can figure out all possibilities which remain at any given time. If there are n possible codewords remaining, the situation is represented in Figure 1 by the notation

n,	if $n \leqslant 2$;
$n(y_1y_2y_3y_4)$,	if $n > 2$ and if the answer to test pattern $y_1y_2y_3y_4$ will uniquely characterize the codeword;
$n(y_1y_2y_3y_4x)$,	if $n > 2$ and if the situation after test pattern $y_1y_2y_3y_4$ is not always unique but there will never be more than 2 possibilities;

$n(y_1y_2y_3y_4: \alpha_{04},\alpha_{03},\alpha_{02},\alpha_{01},\alpha_{00};\alpha_{13},\alpha_{12},\alpha_{11},\alpha_{10};\alpha_{22},\alpha_{21},\alpha_{20};\alpha_{31},\alpha_{30};\alpha_{40})$,
if $n > 2$ and if the test pattern $y_1y_2y_3y_4$ answered by j black hits and k white hits leads to situation α_{jk}.

When $n = 2$, the test pattern doesn't need to be specified, since the best approach is to name either of the two remaining possibilities.

The beginning of Figure 1 can be freely translated as follows:

> There are 1296 possibilities to start with. Your first test pattern should be 1122. Now if the answer is "4 white hits," only one codeword is possible (you can figure it out, it is 2211), so you should guess that code next and win. If the answer is "3 white hits," only 16 possibilities remain, and your next codeword should be 1213; the response must indicate at least one black hit, and if it is, say, "1 black hit, 2 white hits" there are four possibilities which can be distinguished by the answer to test pattern 1415.

Let's consider a more typical play of the game using the strategy of Figure 1. If the answer to the first test pattern 1122 is "one black hit," the original 1296 possibilities have been reduced to 256, and we go to the ninth situation of the fifteen situations following "1122:" in Figure 1; this is the situation just before the second semicolon. The second test pattern (cf. "F") is 1344, and let's say the answer is "one white hit." There are 44 possibilities remaining, and our third guess is 3526. If the result this time is "one black hit, two white hits,"

Figure 1 says that seven possibilities survive, and the test pattern 1462 will distinguish them. For example, the reader might wish to deduce the unique codeword which should be guessed after the following sequence:

test pattern	*hits*
1122	B
1344	W
3526	BWW
1462	BW

(The answer appears at the end of this article.)

Incidentally, the fourth move 1462 in this example is really a brilliant stroke, a crucial play if a win in five is to be guaranteed. None of the seven codewords which satisfy the first three patterns could be successfully used as the fourth test pattern; for example, if we tried 4625, the response WW fails to distinguish 3662 from 5532. A codeword which cannot possibly win in four is necessary here in order to win in five.

Figure 1 was found by choosing at every stage a test pattern that *minimizes the maximum number of remaining possibilities,* over all 15 responses by the codemaker. If this minimum can be achieved by a "valid" pattern (making "4 black hits" possible), a valid one should be used. Subject to this condition, the first such test pattern in numeric order was selected. Fortunately this procedure guarantees a win in five moves.

If the first test pattern is 1123 instead of 1122, the same approach nearly works, but there is one line of play that fails to win in five, starting with

1123	WW	[222 possibilities]
2214	WW	[36 possibilities]
3341		

The 6 possibilities now remaining after "BB" and the 7 after "BBB" cannot all be distinguished by any fourth move. It is probably possible to fix this variation up, by changing the second test pattern; however, when the first test pattern is 1234 it appears to be impossible to guarantee a win in five, especially after 1234 WW [312 possibilities].

The strategy in Figure 1 isn't optimal from the "expected number of moves" standpoint, but it is probably very close to optimal. One line that can be improved occurs in D, after

1122	BWW
1213	BB

There are four possible codewords (2212, 4212, 5212, 6212), and the pattern 4222x distinguishes them more quickly than the arithmetically smallest decent pattern 1145x does.

Figure 1

1296(1122: 1,16(1213: 0,0,0,0,0;1,4(1415),3(1145),0;1,3(4115),3(1145);0,1;0),96A,256B,256C;0,36D,208E,256F;
4(1213),32G,114H;0,20I;1) ,

where

A = (2344: 0,2,16(3215: 0,0,0,0,0;1,2,1,1;2,3(3231),2;0,3(3213);1),
14(5215: 0,0,0,0,0;0,1,3(3511),3(3611);1,1,2;0,2;1),4(1515);0,6(2413),
18(2415: 1,1,0,0,0;1,2,3(2253),3(2236);1,2,2;0,1;1),15(2256x);0,4(2234),14(3315x);0,3(2314);0)

B = (2344: 0,7(2335),41(3235: 0,0,2,3(4613),2;0,3(5263),6(3413),6(3416);2,4(3256),6(1336);0,6(1536);1),
44(3516: 1,4(4651),6(6255),1,0;3(5613),7(1461),5(4551),1;3(1113),5(3551),3(4515);0,4(1145);1),
16(5515: 0,0,1,1,0;0,2,2,1;1,1,3(1516);0,3(1516);1);
2,21(3245: 1,3(2436),0,0,0;2,2,2,0;2,3(3234),2;0,3(3243);1),
42(4514: 1,1,7(2456),4(2635),3(2636);0,4(1356),5(4361),6(1635);2,2,3(3614);0,3(4414);1),
34(3315: 0,0,3(5641),4(2566),1;1,4(5361),4(5614),5(6614);2,4(3331),1;0,4(3316);1);
3(2434),13(2425x),23(1545: 0,1,3(2654),3(2353),4(1136);0,2,4(2564),3(2335);0,0,2;0,1;0);
0,9(1335x);1)

C = (3345: 2,20(4653: 2,2,0,0,0;3(4536),3(4534),1,0;2,2,1;0,3(4453);1),
42(6634: 0,3(4566),4(4556),1,0;2,5(4656),6(5653),4(1444);2,5(5636),5(4654);0,4(1413);1),
16(6646: 0,0,1,0,0;0,3(1416),1,1;3(1416),3(5666),2;0,2;0),1;
4(3453),40(3454: 1,5(4535),6(1436),0,0;2,5(4356),6(3536),0;1,3(3564),6(3463);0,4(3456);1),
46(3636: 1,1,3(4364),6(4565),6(4544);0,5(4366),6(1565),6(4546);2,4(3466),3(3556);0,2;1),
18(3656: 0,1,1,1,1;0,3(5665),3(6446),3(4446);0,1,3(4646);0,1;0);
5(3435x),20(3443: 0,0,4(4355),0,0;0,3(3334),4(3356),0;1,2,4(3455);0,1;1),
29(3636: 0,1,3(5365),4(6445),4(1444);0,2,3(3565),4(4645);1,1,4(3446);0,2;0);0,12(3446x);1)

D = (1213: 1,4(1145),3(1415),0,0;0,6(1114x),7(2412x),0;2,4(1145),4(1145x);0,4(1114x);1)

E = (1134: 0,4(1312),24(3521: 1,2,4(4612),0,0;0,3(3312),3(2423),0;2,2,3(4621);0,3(3321);1),
38(2352: 2,4(3226),4(5621),1,0;1,5(2223),7(6242),1;2,4(2323),4(2462);0,2;1),
20(2525: 1,2,1,0,0;0,3(2252),3(2262),0;2,2,2;0,3(2225);1);
4(1341),34(1315: 1,3(4151),4(4161),0,0;1,6(6451),6(1461),0;3(1351),3(1361),2;0,4(1113);1),
32(1516: 2,2,3(2145),0,4(2324);2,4(1661),4(1245),0;3(1561),3(1551),1;0,3(1511);1),
22(1256: 1,0,4(2524),2,0;0,2,4(5224),4(2224);2,0,0;0,2;1);4(1314),12(1315x),12(1235x);0,2;0)

F = (1344: 0,7(1335),41(3135: 0,0,2,3(4623),2;0,3(5163),6(3423),6(3426);2,4(3156),6(1436);0,6(1536);1),
44(3526: 1,4(4652),6(6155),1,0;3(5623),7(1462),5(4552),1;3(1123),5(3552),3(4525);0,4(1145);1),
16(5525: 0,0,1,1,0;0,2,2,1;1,1,3(1516);0,3(1516);1);
2,21(3145: 1,3(1436),0,0,0;2,2,2,0;2,3(3134),2;0,3(3143);1),
42(4524: 1,1,7(1456),4(1635),3(1636);0,4(1356),5(4362),6(1336);2,2,3(3624);0,3(4424);1),
34(3325: 0,0,3(5642),4(1566),1;1,4(5362),4(5624),5(6624);2,4(3332),1;0,4(3326);1);
3(1434),13(1415x),23(1415: 0,0,2,4(3324),0;0,4(1546),4(1356),4(1136);0,2,3(1136);0,0;0);0,9(1335x);1)

G = (1223: 1,4(2145),3(4115),0,0;0,5(2145),6(4512),0;2,4(1245),3(1415);0,3(1145);1)

H = (1234: 2,16(1325: 1,3(4152),3(4162),0,0;1,3(3126),2,0;1,1,1;0,0;0),
20(1325: 0,3(5162),1,0,0;0,2,4(4522),4(4622);0,3(5125),3(2116);0,0;0),
6(2515),0;4(1323),21(1352: 0,1,2,0,0;2,4(1623),2,0;1,3(1323),3(1462);0,2;1),
16(2156x),12(1315x);2,6(3526),8(1536x);0,1;0)

I = (1223: 0,0,0,0,0;1,5(1145x),4(1114x),0;1,3(1415),4(1114x);0,2;0)

It is not clear what the optimum strategy would be if we use the interesting new scoring rules proposed by G. W. Gill in [4].

Historical Note. A game very similar to Master Mind, called "Bulls and Cows," has been popular in England for many years. The difference is that all digits of the code in Bulls and Cows must be distinct; but any digits 0 through 9 are allowed. This version of the game has become a popular computer demonstration, after Frank H. King introduced a program for it in August 1968 at Cambridge University [1, 5, 6].

References

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Answer to secret codeword: 16 × 227.

ANNOUNCEMENT

New Editors

With this issue the reins of editorship pass from Joe Madachy, who started the Journal, to Harry L. Nelson. Mr. Nelson, often noted in *JRM* as a solver of tough problems, is now taking on another really tough one, that of maintaining the quality while improving the timeliness of the publication. A further statement of editorial policy, and survey of things to come will appear in the next issue. Mr. Madachy will retain a role as a member of the board.

Dave Silverman remains as Problem Editor, and Steve Kahan will edit Alphametics.

The Publisher